Université Grenoble Alpes and ENSIMAG
Master SIAM
Stochastic modelling for neurosciences

Lab 3: Monte Carlo integration

Objectives: *Implementing some Markov Chain Monte Carlo methods.*

1 MCMC for a one dimensional distribution

Let us define the following model:

- $X_0 = -1$, $\phi = -0.1$, $\tau = 1$, $\sigma = 0.2$
- For $k = 1, 2$, $X_{k+1} = \phi X_k + U_k$ with $U_k \sim \mathcal{N}(0, \tau^2)$
- For $k = 0, 1, 2$, $V_k = X_k + \eta_k$ with $\eta_k \sim \mathcal{N}(0, \sigma^2)$.

1. Simulate the variables.
2. Assume we observe X_0, X_2, V_1, V_2 but not X_1. The objective is to filter X_1, ie to estimate the conditional distribution $p(X_1|X_0, X_2, V_1)$.

   This conditional distribution has a density $p(x|X_0, X_2, V_1)$ proportional to

   $$\exp\left[-\frac{1}{2\tau^2}\left((x-\phi X_0)^2 + (X_2 - \phi x)^2 + \frac{\tau^2}{\sigma^2}(V_1 - x)^2\right)\right]$$

   We use a MCMC approach to estimate this distribution, through the implementation of a Random-walk Metropolis-Hastings.

   (a) Consider as proposal q a centered normal distribution with standard deviation $\rho = 0.5$.
   (b) Implement the algorithm, compute the acceptance rate.
   (c) Run the algorithm with 10,000 iterations. From the last 500 realizations of the Markov chain, draw the dynamics of the chain and the density of the chain.
   (d) Change the value $\rho = 0.2$ or $\rho = 1$. Comment.
3. Increase the value of $\sigma = 0.5$ and rerun. Comment.

2 MCMC for filtering a discrete-time process

Let us consider the discretized auto-regressive model for $k = 0, \ldots, n$,

$$X_{k+1} = \phi X_k + U_k$$

with $U_k \sim \mathcal{N}(0, \tau^2)$ and where the observation is

$$V_k = X_k + \eta_k$$

with $\eta_k \sim \mathcal{N}(0, \sigma^2)$.

1. Simulate a trajectory $(V_k, X_k)_{k=0,\dots,n}$ with $\phi = -0.1$, $\tau = 1$, $\sigma = 0.2$, $X_0 = -1$, $n = 100$. Plot the trajectory and the observations.

2. We assume that we observe only the V_k's and the X_k's are hidden. The objective is to filter the hidden process X, ie to estimate the conditional distribution $p(X_{0:n}|V_{0:n})$. We assume that the values of the parameters (ϕ, τ, σ) are known.

   The filtering distribution $p(X_{0:n}|V_{0:n})$ is a multidimensional and complex distribution. Instead of estimating it directly, we split the problem in one dimension problems using the Gibbs algorithm.

   - Initialize the algorithm with $X_{0:n}^{(0)} = V_{0:n} + \epsilon_{0:n}$ where $\epsilon_{0:n}$ is a Gaussian random vector.
   - At iteration $j = 1$ of the Gibbs algorithm, given the current value of the hidden process $X_{0:n}^{(j-1)}$:
     - For $k = 0$, simulate a Markov Chain with stationary distribution equal to the conditional distribution $p(X_0|X_1^{(j-1)}, V_0)$. Set $X_0^{(j)}$ equal to the last value of the chain.
     - For $k = 1, \dots, n-1$, estimate the conditional distribution $p(X_k|X_{k-1}^{(j)}, X_{k+1}^{(j-1)}, V_k)$. Set $X_k^{(j)}$ equal to the last value of the chain.
     - For $k = n$, estimate the conditional distribution $p(X_n|X_{n-1}^{(j)}, V_n)$. Set $X_n^{(j)}$ equal to the last value of the chain.

3. Filter the whole trajectory.