Université Grenoble Alpes and ENSIMAG
Master of Science in Industrial and Applied Math (MSIAM)

Project "Stochastic modelling of neuronal data" ENSIMAG students

Harmonic Oscillator model

Consider the harmonic oscillator driven by a white noise forcing:

$$\begin{cases} dV_t &= U_t dt \\ dU_t &= (-DV_t - \gamma U_t)dt + \sigma dB_t \end{cases} \tag{1}$$

with $\gamma, D, \sigma > 0$. The process has three unknown parameters (D, γ, σ). The drift function and diffusion coefficient are Lipschitz. This is an ergodic Ornstein-Uhlenbeck process, i.e., a Gaussian process.

Some properties

Define

$$X_t = \begin{pmatrix} V_t \\ U_t \end{pmatrix} \quad ; \quad M = \begin{pmatrix} 0 & 1 \\ -D & -\gamma \end{pmatrix} \quad ; \quad \Sigma = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}$$

Then

$$dX_t = MX_t dt + \Sigma dB_t$$

and the conditional distribution is

$$(X_{t+\Delta}|X_t = x) \sim \mathcal{N}\left(e^{\Delta M}x\,,\,\int_0^\Delta e^{sM}\Sigma\Sigma^T e^{sM^T}ds\right).$$

Let $d = \frac{1}{2}\sqrt{\gamma^2 - 4D}$, then

$$\mathbb{E}(X_{t+\Delta}|X_t = x) = e^{-\frac{1}{2}\gamma\Delta}\begin{pmatrix} \left(\cosh(d\Delta) + \frac{\gamma}{2d}\sinh(d\Delta)\right)x_1 + \left(\frac{1}{d}\sinh(d\Delta)\right)x_2 \\ \left(-\frac{D}{d}\sinh(d\Delta)\right)x_1 + \left(\cosh(d\Delta) - \frac{\gamma}{2d}\sinh(d\Delta)\right)x_2 \end{pmatrix}.$$

Making a Taylor expansion in Δ up to order 2 we obtain

$$\mathbb{E}(X_{t+\Delta}|X_t = x) = \begin{pmatrix} x_1 + x_2\Delta - (Dx_1 + \gamma x_2)\frac{\Delta^2}{2} \\ x_2 - (Dx_1 + \gamma x_2)\Delta + (\gamma(Dx_1 + \gamma x_2) - Dx_2)\frac{\Delta^2}{2} \end{pmatrix} + \mathcal{O}(\Delta^3). \tag{2}$$

Furthermore,

$$\begin{aligned} \mathrm{Var}(X_{t+\Delta}|X_t = x) = \frac{\sigma^2}{2\gamma D}\begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} + \\ \frac{\sigma^2 e^{-\gamma\Delta}}{4d^2}\begin{bmatrix} \frac{2}{\gamma} - \frac{d}{D}\sinh(2d\Delta) - \frac{\gamma}{2D}\cosh(2d\Delta) & \cosh(2d\Delta) - 1 \\ \cosh(2d\Delta) - 1 & \frac{2D}{\gamma} + d\sinh(2d\Delta) - \frac{\gamma}{2}\cosh(2d\Delta) \end{bmatrix} \end{aligned} \tag{3}$$

with Taylor expansion up to order 3 in Δ

$$\mathrm{Var}(X_{t+\Delta}|X_t = x) = \sigma^2 \begin{bmatrix} \frac{1}{3}\Delta^3 & \frac{1}{2}\Delta^2 - \frac{1}{2}\Delta^3\gamma \\ \frac{1}{2}\Delta^2 - \frac{1}{2}\Delta^3\gamma & \Delta - \gamma\Delta^2 + \frac{1}{3}\Delta^3(2\gamma^2 - D) \end{bmatrix} + \mathcal{O}(\Delta^4) \tag{4}$$

The invariant distribution is

$$X_\infty \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\gamma D} \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}\right).$$

The solution of this system has thus moments of any order.

Project

In your project, you will study the Harmonic Oscillator (HO) by answering some of the following questions

1. Prove the properties of HO given above
2. Prove that HO is hypoelliptic
3. Simulate some trajectories with an exact scheme or an approximate scheme. Discuss the difference. Typical values are $D = 4, \gamma = 0.5, \sigma = 1$.
4. Study an estimation method of parameters D, γ, σ when both coordinates V, U are observed at discrete times. The estimation method could be
   - Bayesian approach based on Monte Carlo Markov Chain method: Pokern et al 2011
   - Minimization of the contrast defined by the Euler discretization: Samson, Thieullen 2012
   - Minimization of the contrast defined by the local linearization: Léon et al 2016.

   You will first explain the method, give the intuition, resume the principal theoretical results, and try to implement the method on simulated data.
5. Propose a method to filter the unobserved component U when only V is observed at discrete times and when parameters are known or not. Filtering can be based on
   - Kalman filter: Paninski et al 2010
   - Linearization and MCMC: Pokern et al 2011
   - Particle filter: Paninski et al 2012, Ditlevsen and Samson 2014

   You will first explain the method, give the intuition, resume the principal theoretical results if there exist, and try to implement the method on simulated data.
6. Propose an estimation method of parameters D, γ, σ when only V is observed at discrete times. The estimation method could be
   - EM algorithm coupled with a filter: Paninski et al 2010; Paninski et al 2012; Ditlevsen and Samson 2014
   - Bayesian approach based on Monte Carlo Markov Chain method: Pokern et al 2011

   You will first explain the method, give the intuition, resume the principal theoretical results if there exist, and try to implement the method on simulated data.